MEROMORPHIC FUNCTIONS OF ELEMENTS OF A COMMUTATIVE BANACH ALGEBRA

BY

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Introduction. We are concerned here with applications of the author's paper *The Riemann sphere of a commutative Banach algebra* [7] to operational calculus and to abstract analytic function theory. Let A be a commutative complex Banach algebra with identity, A_∞ its Riemann sphere, C the complex plane, and C_∞ the classical Riemann sphere, i.e. the extended complex plane. In §0, for the reader's convenience, we present the relevant theorems and definitions of [7], together with some minor additions. No proofs are given there; they are either in [7], or straightforward.

In §1 $h(p)$ is defined, where $p \in A_\infty$ and h is C_∞-valued and meromorphic on a neighborhood of the spectrum of p in C_∞. (Thus in particular, $h(p)$ is defined when h is a rational function.) The sum $h_1(p) + h_2(p)$ is seen to be defined in A_∞ iff $h_1 + h_2$ have no common pole on the spectrum of p; then $h_1(p) + h_2(p) = (h_1 + h_2)(p)$. Similarly, the product $h_1(p)h_2(p)$ is seen to be defined in A_∞ iff there is no point in the spectrum of p which is simultaneously a pole of one of the h_i and a zero of the other; then $h_1(p)h_2(p) = (h_1h_2)(p)$. The spectral mapping theorem spectrum $h(p) = h(\text{spectrum } p)$ is proved. A straightforward consequence of this operational calculus is that each meromorphic $h: C \to C_\infty$ lifts uniquely to a meromorphic $h: A \to A_\infty$.

The chief result of §2 is a generalization of the Mittag-Leffler theorem. We show that given a sequence z_n in C with no limit point, and a suitable principal part of a Laurent expansion (with coefficients in A) at each z_n, the classical Mittag-Leffler construction yields a meromorphic $\varphi: A \to A_\infty$ so that $\varphi|C$ is A-valued and analytic except at the z_n, $\varphi|C$ has poles at the z_n, the principal part of the Laurent expansion of $\varphi|C$ at each z_n is the one given above, and $\varphi(a)$ lies in A iff the spectrum of a contains none of the z_n. Useful in the proof is the "quotient function lemma" [7, p. 22]. As a corollary, the Mittag-Leffler decomposition of a meromorphic $h: C \to C_\infty$ is seen to remain valid for the meromorphic extension $h: A \to A_\infty$.

To accomplish this generalization, it is necessary to redefine the notion of "pole of an analytic A-valued function of a complex variable." The standard definition [4, p. 236], is "an isolated singularity of f at which the principal part of the Laurent expansion has finitely many nonzero terms." Our definition is "an isolated

singularity of f at z_0 which may be removed by defining $f(z_0)$ to be some point of A_∞ whose spectrum $=\{\infty\}$." Neither of these two notions of pole implies the other for general A: the distinction between them is basically the distinction between the concepts of "polynomial" and "algebraic polynomial." In fact, f has a pole at z_0 (in our sense) iff the principal part of the Laurent expansion of $f(z)$ at z_0 is of the form $\mu((z-z_0)^{-1})$, where μ is an algebraic polynomial satisfying $\mu(\infty)=\infty$.

It is also proved in §2 that an isolated singularity of an analytic (in the sense of Lorch [9]) A-valued function of an A-variable is removable in A, $A \neq C$. This is an analogue of a well-known fact in several complex variables. It was proved by Blum [2] for A in which the invertible elements are dense.

We pose the question: if $h: C \to A_\infty$ is meromorphic, can it be extended to a meromorphic $h: A \to A_\infty$?

BIBLIOGRAPHICAL NOTES. The operational calculus of §1 generalizes that of Taylor [11], who defined $f(T)$, where T is a closed operator, with nonempty resolvent set, on a complex Banach space X, and f is an analytic function defined on a neighborhood of the spectrum of T. In this regard, we note that if λ is in the resolvent of T, then T can be considered to be in the Riemann sphere B_∞ of any commutative Banach algebra B of operators on X which contains $(T-\lambda)^{-1}$ and the identity operator I. But not all elements of the Riemann sphere of an algebra of operators can be interpreted as closed operators. Allan [1] has defined analytic functions of unbounded elements, with nonempty resolvent, of a commutative pseudo-complete locally convex algebra. Gindler [6] has defined $h(T)$ as a closed operator, where T is as above and h is a certain kind of meromorphic function, defined on a neighborhood of the spectrum of T. De Bruijn [3] has defined $h(b)$, where h is a rational function and b is an element of a (not necessarily commutative) Banach algebra with identity B, via a weaker notion of Riemann sphere (see [7, p. 2]). For commutative B, de Bruijn's Riemann sphere is the subset of B_∞ consisting of these q whose spectrum is a proper subset of C_∞.

NOTATION AND TERMINOLOGY. Let S, S_1 and S_2 be sets.

1. If $f: S_1 \to S_2$, and $S \subset S_1$, $f|S=$ the restriction of f to S.

2. If $f: S \to S_1$ and $g: S_1 \to S_2$, then $g \circ f$ denotes the composition of f and g.

Let Y be a topological space.

3. D is a domain in Y iff D is an open connected subset of Y.

4. $C(Y)=$ the Banach algebra of all continuous bounded complex-valued functions on Y with pointwise algebraic operations and the sup norm.

5. $C=$ the complex number field, while $C_\infty=$ the extended complex plane $=$ the Riemann sphere of C.

6. A will always denote a complex commutative Banach algebra with identity.

7. $\mathscr{M}=$ the maximal ideal space of A. As usual, we identify the maximal ideals M of A with the associated multiplicative linear functionals $F: A \to C$.

8. As is usual, we identify the complex number 1 with the identity element of A. Thus C is considered to be a subset of A.

9. z will be used to denote complex numbers and complex variables. Thus, for example, if $f: A \to A$, $\lim_{z \to \infty} f(z)$ will be understood to have the same meaning as $\lim_{z \to \infty} (f|C)(z)$.

10. If $a_0 \in A$ and R is a nonnegative extended real number, $B(a_0 : R)$ denotes the norm ball of radius R about a_0, while $K(a_0 : R)$ denotes the complex disc of radius R about $a_0 =$ the set of all $a_0 + z$, where $|z| < R$.

11. We will have occasion to consider complex-valued analytic functions of a complex variable, A-valued analytic functions of a complex variable, and A-valued analytic functions of an A-variable. When it is necessary to avoid ambiguity, or simply convenient, we shall call functions of the first kind C-analytic, and functions of the third kind A-analytic. An exposition of the theory of A-valued analytic functions of a complex variable may be found in [4, pp. 191–246]. A-analytic functions will be understood to be analytic in the sense of Lorch [9]: [9] is also the prime source for basic information about A-analytic functions. All analytic functions considered here will be single-valued. We remark that the usage of "analytic" here is exactly the same as that of "holomorphic" in [7].

0. Synopsis of the prerequisites from "The Riemann sphere of a commutative Banach algebra." An ordered pair (s, t) of elements of A is called admissible iff there are $\alpha, \beta \in A$ so that $\alpha s + \beta t = 1$, or equivalently, iff there is no maximal ideal F so that $F(s) = F(t) = 0$. Sometimes we shall say, "s and t have no common zero" instead of "(s, t) is admissible." The Riemann sphere A_∞ of A is the set of all admissible pairs, with two pairs (s, t) and (u, v) identified iff $sv = tu$, or equivalently, iff there is an invertible $x \in A$ so that $xu = s$ and $xv = t$. A is identified with a subset of A_∞ via $a \leftrightarrow (a, 1)$, and the classical Riemann sphere C_∞ is identified with a subset of A_∞ (recall that C is already identified with a subset of A) by letting the point at infinity correspond to $(1, 0)$: we will write $\infty = (1, 0)$.

Two admissible pairs (s, t) and (u, v) are defined to have a sum $(s, t) + (u, v) = (sv + tu, tv)$ iff $(sv + tu, tv)$ is admissible: they are defined to have a product $(s, t) \cdot (u, v) = (su, tv)$ iff (su, tv) is admissible. The inverse of any $(s, t) \in A_\infty$ is defined by $(s, t)^{-1} = (t, s)$. (Clearly $(s, t) \cdot (s, t)^{-1}$ is defined in A_∞ iff st is invertible.) It follows that if $s, t \in A$, the product $s \cdot t^{-1}$ is defined in A_∞ (and $= (s, t)$) iff (s, t) is admissible.

We summarize the elementary quasi-algebraic properties of the addition and multiplication of A_∞. (The term "quasi-algebraic" is used because the sum or product of two elements of A_∞ is not always defined.)

THEOREM 0.1. *Let $p, q, r \in A_\infty$, then*

(i) *if $p + q(p \cdot q)$ is defined, then so is $q + p(q \cdot p)$, and $p + q = q + p(p \cdot q = q \cdot p)$,*

(ii) *if $(p + q) + r((p \cdot q) \cdot r)$ is defined, then so is $p + (q + r)(p \cdot (q \cdot r))$ and $(p + q) + r = p + (q + r)((p \cdot q) \cdot r = p \cdot (q \cdot r))$,*

(iii) *$p + (0, 1)(p \cdot (1, 1))$ is defined and $= p$,*

(iv) *if $p \cdot q + p \cdot r$ is defined, then so is $p \cdot (q + r)$, and $p \cdot q + p \cdot r = p \cdot (q + r)$,*

(v) $(p^{-1})^{-1} = p$,

(vi) *if $p \cdot q$ is defined, then so is $p^{-1} \cdot q^{-1}$, and $(p \cdot q)^{-1} = p^{-1} \cdot q^{-1}$,*

(vii) *if $p \in A$ ($p \in A$ and p is invertible in A) then $p + q(p \cdot q)$ is defined.*

Each maximal ideal $F: A \to C$ extends naturally to a mapping $F_*: A_\infty \to C_\infty$ via $F_*(s, t) = (F(s), F(t))$.

THEOREM 0.2. *Let $p, q \in A_\infty$, then*

(i) *if $p + q(p \cdot q)$ is defined in A_∞, and $F \in \mathcal{M}$, then $F_*(p) + F_*(q)(F_*(p) \cdot F_*(q))$ is defined in C_∞, and $F_*(p) + F_*(q) = F_*(p+q)(F_*(p) \cdot F_*(q) = F_*(p \cdot q))$.*

(ii) $F_*(p)^{-1} = F_*(p^{-1})$, *when $F \in \mathcal{M}$, and*

(iii) $p + q(p \cdot q)$ *is defined in A_∞ iff $F_*(p) + F_*(q)(F_*(p) \cdot F_*(q))$ is defined in C_∞ for all maximal ideals F.*

If $p_1, \ldots, p_n$ is a finite sequence of elements of A_∞, and $p_i + p_j(p_i \cdot p_j)$ is defined whenever $i \neq j$, then

$$\sum_{k=1}^{n} p_k = p_1 + \cdots + p_n \left(\prod_{k=1}^{n} p_k = p_1 \cdot \cdots \cdot p_n \right)$$

is defined in A_∞. Thus $\sum_{k=1}^{n} p_k (\prod_{k=1}^{n} p_k)$ is defined iff there is no maximal ideal F and distinct indices i, j so that $F_*(p_i) = F_*(p_j) = \infty$ ($F_*(p_i) = 0$ and $F_*(p_j) = \infty$).

To each 2×2 matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \mu & \delta \end{smallmatrix} \right)$ with coefficients in A and invertible determinant associate the bijection X of A_∞ defined by $X(s, t) = (\alpha s + \beta t, \mu s + \delta t)$. Each such X is called a fractional linear transformation (f.l.t.) of A. The group of fractional linear transformations will be denoted by $G(A)$: its acts transitively on A_∞. The subgroup of $G(A)$ consisting of those X which are associated to 2×2 matrices of complex numbers will be denoted by $G_C(A)$. Obviously $G_C(A)$ is naturally isomorphic to $G(C)$: if $X \in G_C(A)$ we will denote the corresponding element of $G(C)$ by X_C. For purposes of exposition, we will not identify X and X_C; we will regard them as distinct mathematical objects.

A_∞ is given the unique topology so that A is an open subspace, and each f.l.t. X is a homeomorphism of A_∞. If we set D_+ ($D_.$) equal to the subset of $A_\infty \times A_\infty$ consisting of those ordered pairs (p, q) so that $p + q(p \cdot q)$ is defined, then D_+ and $D_.$ are open in $A_\infty \times A_\infty$, and addition and multiplication are continuous maps of D_+ and $D_.$ respectively into A_∞. Thus finite sums and products, when defined, of continuous A_∞-valued functions are continuous.

If $p_1, \ldots, p_n, \ldots$ is an infinite sequence of elements of A_∞, and $p_i + p_j$ is defined whenever $i \neq j$, $\sum_{k=1}^{\infty} p_k$ converges (to $p \in A_\infty$) iff $\lim_{n \to \infty} \sum_{k=1}^{n} p_k$ exists in A_∞ (and equals p). If $q_1, \ldots, q_n, \ldots$ is a sequence of elements of A_∞, and there is $N > 0$ so that $q_n \in A$ when $n > N$ and $\sum_{k=1}^{N} q_k$ is defined in A_∞, and $\sum_{k=N+1}^{\infty} q_k$ is convergent in A, then clearly $\sum_{k=1}^{\infty} q_k$ is convergent in A_∞ to $\sum_{k=1}^{N} q_k + \sum_{k=N+1}^{\infty} q_k$.

If $F \in \mathcal{M}$, $F_*: A_\infty \to C_\infty$ is continuous. The spectrum of $p \in A_\infty$ is defined to be $\sigma(p) = $ the set of all $F_*(p)$ as F ranges over $\mathcal{M}$: $\sigma(p)$ is a compact subset of C_∞.

Maximal ideals distinguish points of A from points of $A_\infty \sim A$: more precisely, if $p \in A_\infty$, then $p \notin A$ iff there is some maximal ideal F so that $F_*(p) = \infty$ iff ∞ is in the spectrum of p. If $X \in G_C(A)$, then a spectral mapping theorem holds: $X(\sigma(p)) = \sigma(X(p))$, all $p \in A_\infty$; in fact $F_*(X(p)) = X(F_*(p))$, all $F \in \mathcal{M}$.

The principal component $A_{\infty p}$ of A_∞ is defined to be the connected component of A_∞ which contains A. (If $A = C(S^3)$, where S^3 is the 3-sphere, $A_{\infty p}$ is a proper subset of A_∞.) If $\sigma(p)$ is a proper subset of C_∞, then $p \in A_{\infty p}$. If $\sigma(p) = C_\infty$, then p may or may not lie in $A_{\infty p}$.

A mapping $f: D \to A_\infty$, where D is a domain in A_∞, is called meromorphic iff for each $q \in D$ there are X and Y in $G(A)$ so that $X(q)$ and $Y(f(q))$ lie in A and $Y \circ f \circ X^{-1}$ is A-analytic at $X(q)$. A mapping $f: D \to A_\infty$, where D is a domain in C_∞, is called meromorphic iff for each $w \in D$ there are X in $G(C)$ and Y in $G(A)$ so that $X(w)$ is in C, $Y(f(w))$ is in A, and $Y \circ f \circ X^{-1}$ is analytic at $X(w)$. Finite sums and products, when defined, inverses, and compositions, when defined, of meromorphic functions are meromorphic. If $f, g: D \to A_\infty$ are meromorphic, where D is a domain in A, and $f = g$ on some $K(a:R) \subseteq D$, then $f = g$ on D. We will adopt a convention for meromorphic functions analogous to that in effect for analytic functions: a meromorphic A_∞-valued (C_∞-valued) function of an A_∞-variable (C_∞-variable) will sometimes be called A-meromorphic (C-meromorphic).

The concepts of "simple algebraic polynomial" and "algebraic polynomial" are fundamental: we will describe them briefly and refer the reader to [7, §§3.3 and 3.5], for detailed definitions. We caution the reader that although the term "polynomial" has its usual meaning in this paper when not preceded by the word "algebraic," neither simple algebraic polynomials nor algebraic polynomials must be polynomials: conversely, there are polynomials which are neither simple algebraic polynomials nor algebraic polynomials.

In algebras A without radical, a simple algebraic polynomial (s. al. p.) f is simply a polynomial with coefficients in A and invertible leading coefficient. f is thought of, via the quasi-algebra of A_∞, as an A_∞-valued function defined on all of A_∞. If A has a nontrivial radical, then composition with a finite number of radical transformations is allowed in the construction of f; a radical transformation being an f.l.t. X associated to a matrix of the form $\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$, where r is in the radical of A. The significant property of radical transformations X is that $X(A) \subset A$; it is reflected in the following important analytic description of simple algebraic polynomials (Theorem 3.3.4 of [7]).

THEOREM 0.3. *An analytic* $g: A \to A$ *is the restriction to* A *of a (unique) s. al. p.* f *iff*

(i) $\lim_{z \to \infty} g(z)$ *exists in* A_∞, *and*

(ii) *if* $g(a) = \sum_{k=1}^{\infty} \alpha_k a^k$ *is the power series expansion of* g *on* A, *then there is some* $N \geq 0$ *so that* α_N *is invertible and* α_k *is in the radical for* $k > N$.

An algebraic polynomial (al. p.) μ is a finite "direct sum" of simple algebraic polynomials. To construct an al. p., take a finite sequence $j_1, \ldots, j_n$ of idempotents

of A so that $\sum_{k=1}^n j_k = 1$ and $j_k j_i = 0$ when $i \neq k$, and s. al. p.'s $f_1, \dots, f_n$ of $j_1 A, \dots,$ $j_n A$ respectively; these naturally induce a meromorphic $\mu: A_\infty \to A_\infty$ which maps A into itself since $(j_1 A)_\infty \times \cdots \times (j_n A)_\infty$ can be identified with A_∞ (see [7, §§2.5 and 3.5]). Such μ are called algebraic polynomials. The analytic description of algebraic polynomials corresponding to 0.3 is

THEOREM 0.4. *An analytic $g: A \to A$ is the restriction to A of a (unique) algebraic polynomial μ iff* $\lim_{z \to \infty} g(z)$ *exists in A_∞.*

Finally, we state the "quotient function" lemma.

LEMMA 0.5. *If $h: A_{\infty p} \to A_{\infty p}$ is meromorphic, and F is a maximal ideal, then there is a unique C-meromorphic $h_F: C_\infty \to C_\infty$ so that $h_F(F_*(q)) = F_*(h(q))$, all $q \in A_{\infty p}$. h_F is called the quotient function of h with respect to F.*

1. **C-meromorphic functions of $p \in A_\infty$.** For each $p \in A_\infty$, let $\mathscr{A}(p)$ be the algebra of C-analytic functions defined on an open neighborhood in C_∞ of the spectrum of p, and let $Q(p)$ be the field of C-meromorphic functions defined on an open neighborhood in C_∞ of the spectrum of p; in both cases two functions are identified if they agree on a C_∞-neighborhood of $\sigma(p)$. Let Δ denote the field of complex rational functions, i.e. quotients of polynomials with complex coefficients. Note that when the spectrum of p is C_∞, $\Delta = Q(p)$.

The standard operational calculus (s.o.c.) defines $f(p)$ where $f \in \mathscr{A}(p)$ and $p \in A$, by

$$f(p) = (2\pi i)^{-1} \int_\Gamma f(z)(z-p)^{-1} \, dz,$$

where Γ is a suitable system of simple closed rectifiable curves enclosing the spectrum of p in C. (For a detailed description of Γ, see e.g. Hille and Phillips [8, p. 166].) We single out two properties of the s.o.c.: that for each fixed $p \in A$, the mapping $f \to f(p)$ is a complex algebra with identity homomorphism of $\mathscr{A}(p)$ into A which sends $f(z) = z$ into p, and that $\sigma(f(p)) = f(\sigma(p))$, all $f \in \mathscr{A}(p)$ and $p \in A$.

To define $h(p)$, where $h \in Q(p)$ and $p \in A_\infty$, we will consider three cases: when $h \in \mathscr{A}(p)$ and $\sigma(p)$ is a proper subset of C_∞, when $h \in Q(p)$ and $\sigma(p)$ is a proper subset of C_∞, and when $h \in \Delta$ and p is an arbitrary element of A_∞. In each of these cases we will establish analogues, as far as is possible, of the two properties of the s.o.c. mentioned above.

REMARK. Let $X \in G_C(A)$ and $p \in A$, and suppose that X_C is in $\mathscr{A}(p)$. Then it follows directly from the s.o.c. that $X_C(p) = X(p)$.

1.1. *C-analytic functions of $p \in A_\infty$, when $\sigma(p)$ is a proper subset of C_∞.* In this section, p will denote an element of A_∞ whose spectrum is a proper subset of C_∞, and f will denote a function in $\mathscr{A}(p)$.

We define $f(p)$ as follows: choose a fractional linear transformation $X \in G_C(A)$ so that $\infty \notin X(\sigma(p))$. Note that by the spectral mapping theorem of §0, $\infty \notin \sigma(X(p))$,

so $X(p)$ is in A. Now set

$$f(p) = (f \circ X_C^{-1})(X(p)) \in A$$

where the right-hand side is defined by the standard operational calculus. (This device was introduced by Taylor in [11] to take analytic functions of unbounded operators.)

To see that the above definition is independent of the choice of X, let Y be another element of $G_C(A)$ so that ∞ is not in $Y(\sigma(p))$. Write $a = X(p)$ and $Z = Y \circ X^{-1}$, then $Z(a) = Y(p) \in A$. Set $g = f \circ X_C^{-1} \in \mathscr{A}(a)$. The standard operational calculus (Dunford and Schwartz [5, p. 570]) shows that

$$g(a) = (g \circ Z_C^{-1})(Z_C(a)).$$

Therefore

$$(f \circ X_C^{-1})(X(p)) = g(a) = (g \circ Z_C^{-1})(Z_C(a)) = (g \circ Z_C^{-1})(Z(a)) = (f \circ Y_C^{-1})(Y(p)).$$

Note that when $p \in A$, X can be chosen to be the identity f.l.t. Thus the definition of $f(p)$ given here is consistent with the standard one for $p \in A$.

Theorem 1.1.1. *If X is any f.l.t. of $G_C(A)$, then $(f \circ X_C^{-1})(X(p)) = f(p)$.*

Proof. Choose $Y \in G_C(A)$ so that $Y(p) \in A$, and set $Z = Y \circ X^{-1}$. Since $Z(X(p)) = Y(p) \in A$,

$$(f \circ X_C^{-1})(X(p)) = (f \circ X_C^{-1} \circ Z_C^{-1})(Z(X(p))) = (f \circ Y_C^{-1})(Y(p)) = f(p).$$

Corollary 1.1.2 *If X is any f.l.t. of $G_C(A)$ so that $X_C \in \mathscr{A}(p)$, then $X_C(p) = X(p)$.*

Proof. Set $f = X_C$ in 1.1.1 to obtain $g(X(p)) = X_C(p)$, where $g(z) \equiv z$. Since $X_C \in \mathscr{A}(p)$, $X(p) \in A$, thus by the s.o.c. $g(X(p)) = X(p)$.

Theorem 1.1.3. *For fixed p, the mapping of $\mathscr{A}(p)$ into A defined by $f \to f(p)$ is a complex algebra with identity homomorphism. When $f(z) \equiv z$, $f(p) = p$.*

Proof. Direct from the s.o.c. and 1.1.2.

Theorem 1.1.4. *If F is a maximal ideal, then $F(f(p)) = f(F_*(p))$. Thus $\sigma(f(p)) = f(\sigma(p))$.*

Proof. If $X \in G_C(A)$ so that $X(p) \in A$, then

$$
\begin{aligned}
F(f(p)) &= F((f \circ X_C^{-1})(X(p))) \\
&= \text{(by the s.o.c.) } (f \circ X_C^{-1})(F(X(p))) \\
&= \text{(by §0) } (f \circ X_C^{-1})(X(F_*(p))) \\
&= f(F_*(p)).
\end{aligned}
$$

$\sigma(f(p)) = f(\sigma(p))$ follows immediately from the definition of spectrum.

1.2. *C-meromorphic functions of $p \in A_\infty$, when $\sigma(p)$ is a proper subset of C_∞.* In this section, p will denote an element of A_∞ whose spectrum is a proper subset of C_∞, and h will denote a function in $Q(p)$.

We define $h(p)$ as follows: write (Rudin [10, p. 296]) h as the quotient f/g of two $\mathscr{A}(p)$ functions f and g so that there is no $z \in \sigma(p)$ at which $f(z) = g(z) = 0$. Via 1.1, $f(p)$ and $g(p)$ are well-defined elements of A. Furthermore, by 1.1.4 and the preceding condition on f and g, $f(p)$ and $g(p)$ have no common zero. Set

$$h(p) = f(p)g(p)^{-1} = (f(p), g(p)) \in A_\infty.$$

To see that $h(p)$ is well-defined, suppose that f_1 and g_1 are alternate choices of f and g. Since $f/g = f_1/g_1$, $fg_1 = f_1g$ so, by 1.1.3, $f(p)g_1(p) = f_1(p)g(p)$. But this is precisely what $f(p)g(p)^{-1} = f_1(p)g_1(p)^{-1}$ means.

If $h \in \mathscr{A}(p)$, the above definition of $h(p)$ is clearly consistent with that of $h(p)$ provided in 1.1.

REMARK. Let $h: C \to C_\infty$ be meromorphic, and consider the mapping from A to A_∞ defined by $a \to h(a)$. This mapping is meromorphic on A since the quotient in A_∞ of analytic functions is meromorphic, and is the unique meromorphic extension of h to A since a meromorphic function: $A \to A_\infty$ is determined by its values on C.

THEOREM 1.2.1. *If $X \in G_C(A)$, then $(h \circ X_C^{-1})(X(p)) = h(p)$.*

Proof. Direct from 1.1.1.

COROLLARY 1.2.2. *If $X \in G_C(A)$, then $X_C(p) = X(p)$.*

Proof. A direct computation suffices when $p \in A$. When $p \notin A$, since $\sigma(p)$ is a proper subset of C_∞, we can choose $Y \in G_C(A)$ so that $Y(p) \in A$. 1.2.1 shows that

$$(X_C \circ Y_C^{-1})(Y(p)) = X_C(p),$$

but

$$(X_C \circ Y_C^{-1})(Y(p)) = (X \circ Y^{-1})_C(Y(p))$$
$$= (\text{since } Y(p) \in A) \ (X \circ Y^{-1})(Y(p)) = X(p).$$

THEOREM 1.2.3. *If F is a maximal ideal, then $F_*(h(p)) = h(F_*(p))$. Thus $\sigma(h(p)) = h(\sigma(p))$.*

Proof. Direct from 1.1.4 and the definition of F_*.

Because we cannot always add and multiply elements of A_∞, the mapping $h \to h(p)$, for fixed p, is not a homomorphism but a "pseudo-homomorphism": i.e. in place of 1.1.3 we have

THEOREM 1.2.4. *Let h_1 and h_2 be in $Q(p)$. Then $h_1(p) + h_2(p)$ is defined in A_∞ iff $h_1(z) + h_2(z)$ is defined in C_∞ for each z in the spectrum of p; in that case $(h_1 + h_2)(p) = h_1(p) + h_2(p)$. Similarly, $h_1(p)h_2(p)$ is defined in A_∞ iff $h_1(z)h_2(z)$ is defined in C_∞ for each z in $\sigma(p)$; then $h_1(p)h_2(p) = (h_1h_2)(p)$. If $h(z) \equiv 1$, $h(p) = 1$: if $h(z) \equiv z$, $h(p) = p$.*

Proof. That the above conditions for $h_1(p) + h_2(p)$ or $h_1(p)h_2(p)$ to be defined are necessary and sufficient follows immediately from 0.2 (iii) and 1.2.3.

If $h_1(p) + h_2(p)$ is defined, a direct computation via 1.1.3 shows that $h_1(p) + h_2(p) = (h_1 + h_2)(p)$; the corresponding multiplicative result is equally straightforward. If $h(z) \equiv 1$, $h(p) = 1$ follows from 1.3. If $h(z) \equiv z$, $h(p) = p$ is a special case of 1.2.2.

1.3. *Complex rational functions of* $p \in A_\infty$. In this section, p will denote an arbitrary element of A_∞, and h will denote a function in Δ, i.e. a complex rational function. The definition of $h(p)$ will proceed independently of the results established in 1.1 and 1.2.

If h is identically zero, set $h(p) = 0$. If h is not identically zero, write $h = f/g$, where f and g are complex polynomials with no roots in common, and the leading coefficient of g is 1. Write

$$f(z) = \lambda \prod_{i=1}^{m} (z - w_i) \quad \text{and} \quad g(z) = \prod_{j=1}^{n} (z - u_j),$$

where $\lambda \neq 0$ and $w_i \neq u_j$, all i, j. We will adopt the convention that all empty products and 0th powers are equal to 1.

Now the quasi-algebra of A_∞ makes possible the definitions

$$f(p) = \lambda \prod_{i=1}^{m} (p - w_i) \quad \text{and} \quad g(p) = \prod_{j=1}^{n} (p - u_j).$$

However, we cannot set $h(p) = f(p)g(p)^{-1}$: for example, if m and n are positive, $f(\infty) = g(\infty) = \infty_1$ and $\infty \cdot \infty^{-1}$. This difficulty can be surmounted.

Define $\bar{f}$ and $\bar{g}$: $A \times A \to A$ by

$$\bar{f}(a, b) = \lambda \prod_{i=1}^{m} (a - w_i b) \quad \text{and} \quad \bar{g}(a, b) = \prod_{j=1}^{n} (a - u_j b).$$

$\bar{f}$ and $\bar{g}$ are homogeneous of degree m and n respectively, i.e.

$$\bar{f}(xa, xb) = x^m \bar{f}(a, b) \quad \text{and} \quad \bar{g}(xa, xb) = x^n \bar{g}(a, b), \quad \text{all } x \in A.$$

Notice also that $\bar{f}(a, 1) = f(a)$ and $\bar{g}(a, 1) = g(a)$.

Write $p = \alpha \beta^{-1}$, where $\alpha, \beta \in A$, and α and β have no common zero. Then

$$f(p) = \bar{f}(\alpha, \beta)(\beta^m)^{-1} \quad \text{and} \quad g(p) = \bar{g}(\alpha, \beta)(\beta^n)^{-1},$$

where the multiplications and inverses used are defined in A_∞. From now on all sums, products and inverses will be considered to be A_∞-sums, products and inverses.

We define $h(p)$ by cases:

$$\text{if } m \geqq n, \quad h(p) = \bar{f}(\alpha, \beta)(\beta^{m-n}\bar{g}(\alpha, \beta))^{-1},$$

and

$$\text{if } m \leqq n, \quad h(p) = \beta^{n-m}\bar{f}(\alpha, \beta)(\bar{g}(\alpha, \beta))^{-1}.$$

We will show that $h(p)$ is well-defined when $m \geqq n$: a similar argument applies when $n \geqq m$. There are two things to check: that $\bar{f}(\alpha, \beta)$ and $\beta^{m-n}\bar{g}(\alpha, \beta)$ have no common zero, so that $\bar{f}(\alpha, \beta)(\beta^{m-n}\bar{g}(\alpha, \beta))^{-1}$ is defined in A_∞, and that $h(p)$ is independent of the choice of α and β.

Suppose F is a maximal ideal so that

$$F(\bar{f}(\alpha, \beta)) = F(\beta^{m-n}\bar{g}(\alpha, \beta)) = 0.$$

If $F(\beta) = 0$, then since $F(\bar{f}(\alpha, \beta)) = 0$, $F(\alpha) = 0$, which contradicts α and β having no common zero. If $F(\beta) \neq 0$, then $f(F(\alpha)/F(\beta)) = g(F(\alpha)/F(\beta)) = 0$, which contradicts the choice of f and g. Thus there is no such F, i.e. $\bar{f}(\alpha, \beta)$ and $\beta^{m-n}\bar{g}(\alpha, \beta)$ have no common zero.

If p also $= \lambda\delta^{-1}$, where $\lambda, \delta \in A$ and λ and δ have no common zero, choose an invertible $x \in A$ so that $x\lambda = \alpha$ and $x\delta = \beta$. Then a straightforward computation via the homogeneity of $\bar{f}$ and $\bar{g}$ shows that

$$\bar{f}(\alpha, \beta)(\beta^{m-n}\bar{g}(\alpha, \beta))^{-1} = \bar{f}(\lambda, \delta)(\delta^{m-n}\bar{g}(\lambda, \delta))^{-1};$$

thus $h(p)$ is well defined.

Observe that one can show by a direct argument in each of the cases $m \geqq n$ and $n \geqq m$ that if $h \neq 0$, then $h(p)^{-1} = h^{-1}(p)$.

THEOREM 1.3.1. *If $X \in G_C(A)$, then*

$$(1) \qquad\qquad h(p) = (h \circ X_C^{-1})(X(p)).$$

Proof. When X_C is a translation, or multiplication by a nonzero complex number, a direct verification yields (1). When $X_C(z) = 1/z$ and $m \geqq n$, (1) follows from a straightforward computation in the separate cases $h(0) = 0$, $h(0) = \infty$, and $h(0) = w$, where $w \in C$ and $w \neq 0$. The case $X_C(z) = 1/z$ and $n \geqq m$ is obtained from the preceding case and taking inverses.

1.3.1 now follows from the fact that any fractional linear transformation in $G(C)$ can be written as a composition of translations, multiplications by a nonzero scalar, and inversions.

COROLLARY 1.3.2. *The value assigned to $h(p)$ in this section (1.3) is the same as that assigned in 1.2 to $h(p)$, considering h to be in $Q(p)$, when the spectrum of p is a proper subset of C_∞.*

Proof. The consistency of the two values of $h(p)$ is immediate when $p \in A$. When $p \notin A$, since $\sigma(p) \neq C_\infty$ we can choose $X \in G_C(A)$ so that $X(p) \in A$. 1.3.2 now follows from 1.2.1, 1.3.1 and the preceding case.

COROLLARY 1.3.3. *If $X \in G_C(A)$, $X_C(p) = X(p)$.*

Proof. Take $h = X_C$ in 1.3.1.

THEOREM 1.3.4. *If F is a maximal ideal, then $F_*(h(p)) = h(F_*(p))$. Thus $\sigma(h(p)) = h(\sigma(p))$.*

Proof. By a direct computation in each of the cases $m > n$ and $F(\beta) = 0$, $m \geq n$ and $F(\beta) \neq 0$, and $m = n$ and $F(\beta) = 0$. The case $m < n$ can be reduced to the case $n < m$ by taking inverses.

Now we will show that for fixed p, the mapping $h \to h(p)$ is a "pseudo-homomorphism," just as in 1.2.4. Here it will be helpful to consider addition and multiplication separately.

THEOREM 1.3.5. *Let h_1 and $h_2 \in \Delta$. $h_1(p)h_2(p)$ is defined in A_∞ iff $h_1(z)h_2(z)$ is defined in C_∞ for each $z \in \sigma(p)$: then $h_1(p)h_2(p) = (h_1h_2)(p)$.*

Proof. That $h_1(p)h_2(p)$ is defined in A_∞ iff $h_1(z)h_2(z)$ is defined in C_∞ for each $z \in \sigma(p)$ follows immediately from 0.2 (iii) and 1.3.4. Now suppose that $h_1(p)h_2(p)$ is defined. By 1.2.4 and 1.3.2 we only need consider the case $\sigma(p) = C_\infty$: thus we may assume that $h_1(z)h_2(z)$ is defined in C_∞ for each $z \in C_\infty$. Clearly we may also assume that neither h_1 nor h_2 is identically zero.

For $k = 1, 2$ write $h_k = f_k/g_k$, where the f_k and g_k are complex polynomials with no common roots, and the leading coefficient of g_k is 1: set $m_k = \text{degree } f_k$ and $n_k = \text{degree } g_k$. Thus $h_1h_2 = f_1f_2/g_1g_2$, where because $h_1(z)h_2(z)$ is always defined f_1f_2 and g_1g_2 have no common roots. We now argue by cases.

If $m_1 > n_1$, then $h_1(\infty) = \infty$; since $h_1(\infty)h_2(\infty)$ is defined, $h_2(\infty) \neq 0$, so $m_2 \geq n_2$. Now a trivial computation shows that $h_1(p)h_2(p) = (h_1h_2)(p)$.

The case $m_1 < n_1$ follows from considering the inverses of h_1 and h_2 and applying the preceding case. The case $m_1 = n_1$ and $m_2 = n_2$ is immediate. The remaining cases follow by the symmetry of 1.3.5 in h_1 and h_2.

THEOREM 1.3.6. *Let h_1 and $h_2 \in \Delta$. $h_1(p) + h_2(p)$ is defined in A_∞ iff $h_1(z) + h_2(z)$ is defined in C_∞ for each $z \in \sigma(p)$; then $(h_1 + h_2)(p) = h_1(p) + h_2(p)$.*

Proof. We can certainly assume that neither h_1 nor h_2 is identically zero. That $h_1(p) + h_2(p)$ is defined iff $h_1(z) + h_2(z)$ is defined for each $z \in \sigma(p)$ follows immediately from 0.2 (iii) and 1.3.4.

Now assume that $h_1(p) + h_2(p)$ is defined. By 1.2.4 and 1.3.2 it again suffices to consider only the case when the spectrum of p is C_∞. Note that if $h_1 + h_2$ is identically zero, then neither h_1 nor h_2 can assume the value ∞: thus by Liouville's theorem h_1 and h_2 are identically constant. So we may also assume without loss of generality that $h_1 + h_2$ is not identically zero.

For $k = 1, 2$ let f_k, g_k, m_k and n_k be as in the proof of 1.3.5. Set $f_3 = f_1g_2 + f_2g_1$, $g_3 = g_1g_2$, $m_3 = \text{degree } f_3$ and $n_3 = \text{degree } g_3 = n_1 + n_2$. Since $h_1(z) + h_2(z)$ is defined, all $z \in C_\infty$, f_3 and g_3 have no common root, so we can write $h_3 = h_1 + h_2 = f_3/g_3$. Now we argue by cases.

Case 1. $m_1 \geq n_1$ and $m_2 \leq n_2$. Here

$$h_1(p) + h_2(p)$$
$$= (\bar{f}_1(\alpha, \beta)\bar{g}_2(\alpha, \beta) + \beta^{m_1 + n_2 - m_2 - n_1}\bar{f}_2(\alpha, \beta)\bar{g}_1(\alpha, \beta)) \cdot (\beta^{m_1 - n_1}\bar{g}_1(\alpha, \beta)\bar{g}_2(\alpha, \beta))^{-1}.$$

Observe that if λ and w are finite complex numbers, with $w \neq 0$, then by the homogeneity of the f_k and g_k, $k = 1, 2$,

$$\bar{f}_1(\lambda, w)\bar{g}_2(\lambda, w) + w^{m_1 + n_2 - m_2 - n_1}\bar{f}_2(\lambda, w)\bar{g}_1(\lambda, w) = w^{m_1 + n_2 - m_3}\bar{f}_3(\lambda, w).$$

Since both sides of the above equation are polynomials in λ and w,

$$\bar{f}_1(a, b)\bar{g}_2(a, b) + b^{m_1 + n_2 - m_2 - n_1}\bar{f}_2(a, b)\bar{g}_1(a, b) = b^{m_1 + n_2 - m_3}\bar{f}_3(a, b), \quad \text{all } a, b \in A.$$

Thus $h_1(p) + h_2(p) = (\beta^{m_1 + n_2 - m_3}\bar{f}_3(\alpha, \beta))(\beta^{m_1 - n_1}\bar{g}_3(\alpha, \beta))^{-1}$. If $m_1 > n_1$, then $m_3 = m_1 + n_2$, so

$$h_1(p) + h_2(p) = \bar{f}_3(\alpha, \beta)(\beta^{m_1 - n_1}\bar{g}_3(\alpha, \beta))^{-1} = h_3(p).$$

If $m_1 = n_1$ then, since $n_3 = n_1 + n_2 = \max(m_1 + n_2, m_2 + n_1) \geq m_3$,

$$h_1(p) + h_2(p) = \beta^{n_3 - m_3}\bar{f}_3(\alpha, \beta)(\bar{g}_3(\alpha, \beta))^{-1} = (h_1 + h_2)(p).$$

Case 2. $m_1 < n_1$ and $m_2 < n_2$. Here

$$h_k(p) = \beta^{n_k - m_k}\bar{f}_k(\alpha, \beta)(\bar{g}_k(\alpha, \beta))^{-1}, \quad k = 1, 2,$$

so

$$h_1(p) + h_2(p) = (\beta^{n_1 - m_1}\bar{f}_1(\alpha, \beta)\bar{g}_2(\alpha, \beta) + \beta^{n_2 - m_2}\bar{f}_2(\alpha, \beta)\bar{g}_1(\alpha, \beta)) \cdot (\bar{g}_1(\alpha, \beta)\bar{g}_2(\alpha, \beta))^{-1}.$$

As in Case 1, show that

$$b^{n_1 - m_1}\bar{f}_1(a, b)\bar{g}_2(a, b) + b^{n_2 - m_2}\bar{f}_2(a, b)\bar{g}_1(a, b) = b^{n_1 + n_2 - m_3}\bar{f}_3(a, b), \quad \text{all } a, b \in A.$$

Thus $h_1(p) + h_2(p) = \beta^{n_3 - m_3}\bar{f}_3(\alpha, \beta)(\bar{g}_3(\alpha, \beta))^{-1}$, which is easily seen to be $(h_1 + h_2)(p)$.

The case $m_2 \geq n_2$ and $m_1 \leq n_1$ is symmetric with Case 1, and the case $m_1 > n_1$ and $m_2 > n_2$ cannot occur, as then $h_1(\infty) + h_2(\infty)$ would be $\infty + \infty$, which is undefined. Thus 1.3.6 is proved.

REMARK. It is immediate from the definition of $h(p)$ that $h(p) = 1$ when $h(z) \equiv 1$, and $h(p) = p$ when $h(z) \equiv z$.

2. Isolated singularities, poles and the Mittag-Leffler theorem.

The extension of A to A_∞ offers a new perspective on isolated singularities of analytic A-valued functions of a complex variable. Suppose that $f: D \to A$, where D is a C-domain, is analytic, and has an isolated singularity at $z_0 \in C$ (i.e. for suitably small $\delta > 0$, $D \supset K(z_0: \delta) \sim \{z_0\}$ but $z_0 \notin D$). The isolated singularity is said to be *removable* iff f can be extended to an analytic $f: D \cup \{z_0\} \to A$: just as in the classical case one proves that the isolated singularity is removable iff f is bounded on some $K(z_0: \delta) \sim \{z_0\}$. The isolated singularity is said to be A_∞-*removable* iff f can be extended to a meromorphic $f: D \cup \{z_0\} \to A_\infty$; it is said to be a *pole* iff it is A_∞-removable and the spectrum of $f(z_0)$ contains only the point at infinity.

Now it is obvious that the singularity of f at z_0 is removable iff for suitably small $\delta > 0$, $g = f | K(z_0: \delta) \sim \{z_0\}$ can be extended to an A-analytic $g: B(z_0: \delta) \to A$. An analogous result for A_∞-removable singularities follows from 0.4; the singu-

larity of f at z_0 is A_∞-removable iff for suitably small $\delta > 0$, $g = f | K(z_0:\delta) \sim \{z_0\}$ can be extended to an A-meromorphic $g: B(z_0:\delta) \to A_\infty$. To see this, suppose that the singularity at z_0 is A_∞-removable, and let $\sum_{n=-\infty}^{\infty} \alpha_n(z-z_0)^n = g(z)$ be the Laurent expansion of g about z_0; set $g_1(z) = \sum_{n=-\infty}^{-1} \alpha_n(z-z_0)^n =$ the principal part of the Laurent expansion of g. Then $\lim_{z \to z_0} g_1(z)$ exists in A_∞, i.e. $\lim_{z \to \infty} \sum \alpha_n z^n$ exists in A_∞. By 0.4 there is an algebraic polynomial $\mu: A_\infty \to A_\infty$ so that $\mu(z) = \sum \alpha_n z^n$, all $z \in C$. Thus

$$g(a) = \sum_{n=0}^{\infty} \alpha_n(a-z_0)^n + \mu((a-z_0)^{-1})$$

provides the desired extension of g. Note that the singularity is a pole iff $\mu(\infty) = \infty$: such algebraic polynomials are called constant-free (because none of the simple algebraic polynomials from which they are constructed are identically constant).

The preceding local discussion was a prelude to the following global result.

THEOREM 2.1 (MITTAG-LEFFLER THEOREM). *Let $z_1, \ldots, z_n, \ldots$ be a finite or infinite sequence of distinct complex numbers with no limit point in the finite plane: associate to each z_n a constant-free algebraic polynomial μ_n so that $\mu_n(0) = 0$. Then there is an A-meromorphic $h: A \to A_\infty$ so that*

  (i) *$h(a) \in A$ iff the spectrum of a contains no z_n, all $a \in A$, and*

  (ii) *the principal part of the Laurent expansion of $h | C$ about each z_n is $\mu_n((z-z_n)^{-1})$. h is unique up to the addition of an analytic $g: A \to A$.*

Proof. Just as in the classical case, it is sufficient to consider only the case when no $z_n = 0$. For each n, since $\mu_n((z-z_n)^{-1})$ is analytic on $K(0:|z_n|)$, choose a polynomial q_n with coefficients in A so that

$$\|\mu_n((z-z_n)^{-1}) - q_n(z)\| \leqq 2^{-n}$$

when $z \in C$ and $|z| \leqq |z_n|/2$. Define $h_n: A \to A_\infty$ by

$$h_n(a) = \mu_n((a-z_n)^{-1}) - q_n(a).$$

h_n is A-meromorphic on A, and A-analytic on $B(0:|z_n|)$. Furthermore $\|h_n(z)\| \leqq 2^{-n}$ when $|z| \leqq |z_n|/2$.

Let F be a maximal ideal of A: by 0.5 for each n there is a unique C-meromorphic $\mu_{nF}: C_\infty \to C_\infty$ so that when p is in the principal component $A_{\infty p}$ of A_∞, we have

(1) $$\mu_{nF} \circ F_*(p) = F_* \circ \mu_n(p).$$

Since an algebraic polynomial maps A into itself, we see by substituting $p = z \in C$ in (1) that μ_{nF} maps C into itself. Since $\mu_n(\infty) = \infty$, substituting $p = \infty$ in (1) shows that $\mu_n(\infty) = \infty$. It now follows from (1) that, when $p \in A_{\infty p}$, $F_*(\mu_n(p))$ is finite iff $F_*(p)$ is. Thus for $a \in A$, $F_*(h_n(a))$ is finite iff $F(a) \neq z_n$. Since maximal ideals distinguish points of A from points of $A_\infty \sim A$, $h_n(a) \in A$ iff z_n is not in the spectrum of a, for all $a \in A$ and positive integers n.

For an arbitrary positive integer N, choose a positive integer I so that $|z_n| \geq 2N$ when $n \geq I$. Then $\sum_{n=I}^{\infty} h_n(z)$ converges uniformly on $K(0:N)$ to an analytic A-valued function of a complex variable. It now follows from the Cauchy integral formula for A-analytic functions that $\sum_{n=I}^{\infty} h_n(a)$ converges on $B(0:N)$ to an A-analytic function.

But $\sum_{n=1}^{I-1} h_n(a)$ is defined in A_∞ for all $a \in A$, since (see §0) if $i \neq j$ and $F_*(h_i(a)) = F_*(h_j(a)) = \infty$, $F(a) = z_i = z_j$, which contradicts the assumption of distinct z_n. Thus $\sum_{n=1}^{\infty} h_n(a)$ converges on $B(0:N)$, and therefore on A to an A-meromorphic function $h(a)$. If F is a maximal ideal, clearly $F_*(h(a)) = \infty$ iff for some n, $F_*(h_n(a)) = \infty$ iff, for some n, $F(a) = z_n$. Thus $h(a) \in A$ iff the spectrum of a contains no z_n, all $a \in A$.

Now the principal parts of the Laurent expansions of $h|C$, $h_n|C$ and $\mu_n((z - z_n)^{-1})$ about z_n are easily seen to coincide. Since $\mu_n(0) = 0$ and $\mu_n(z)$ is analytic on C, $\mu_n((z - z_n)^{-1})$ is equal to the principal part of its Laurent expansion about z_n.

For uniqueness, suppose that the meromorphic $\varphi: A \to A_\infty$ also satisfies (i) and (ii). Then by (i) we can define $g: C \sim \{z_n\}_{n=1}^{\infty} \to A$ by $g(z) = \varphi(z) - h(z)$. By (ii), g has a removable singularity at each z_n, thus g can be extended to an analytic $g: C \to A$. The power series expansion of g can now be used to lift g to an analytic $g: A \to A$ so that $h(a) + g(a) = \varphi(a)$, all $a \in A$.   Q.E.D.

In the proof of 2.1, we took care to require only that the q_n approximate the $\mu_n((z - z_n)^{-1})$ on suitable subsets of C. Thus if the μ_n are complex polynomials, the classical construction of a C-meromorphic function on C whose poles are the z_n and whose principal parts at the z_n are the $\mu_n((z - z_n)^{-1})$ lifts completely to the construction of an A-meromorphic function defined on all of A. (Any complex polynomial can be thought of, via the quasi-algebra of A_∞, as a simple algebraic polynomial and thus an algebraic polynomial of A.)

REMARK. We have used the Riemann sphere A_∞ to discuss isolated singularities of an analytic A-valued function of a complex variable. The situation for analytic A-valued functions of an A-variable $(A \neq C)$ is quite different: there each "isolated singularity" is "A-removable". More precisely, each A-analytic function defined on a punctured ball $B(a_0:R) \sim \{a_0\}$, $a_0 \in A$, can be extended to an A-analytic function defined on $B(a_0:R)$.

To see this, consider without loss of generality an A-analytic $f: B(0:1) \sim \{0\} \to A$. Choose $b_0 \in A \sim C$ so that $\|b_0\| = 1/4$. $f(a)$ has a convergent Taylor expansion $\sum \alpha_n(a - b_0)^n$, valid for all a in some ball $B(b_0:\delta)$ about b_0. Since

$$K(b_0:1/2) \subset B(0:1) \sim \{0\},$$

$\sum \alpha_n z^n$ converges to $f(b_0 + z)$, all $z \in K(b_0:1/2)$. Define $f_1: B(b_0:1/2) \to A$ by $f_1(a) = \sum \alpha_n(a - b_0)^n$. f_1 is A-analytic on $B(b_0:1/2)$ and agrees with f on $K(b_0:1/2)$, thus f_1 and f agree on $B(b_0:1/2) \sim \{0\}$. Therefore f can be analytically extended to $B(0:1)$ by defining $f(0) = f_1(0)$.

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